RUHUNA JOURNAL OF SCIENCE
Vol. II, September 2007, pp. 10–17
http://www.ruh.ac.lk/rjs/
ISSN 1800-279X

A solution for non-stationary, slowly-rotating, cylindrically symmetric, perfect fluid universe

Wijewickrema P.K.C.M. and Wijayasiri M.P.A.
Department of Mathematics, University of Ruhuna, Matara, Sri Lanka.
Correspondece: wijaya@maths.ruh.ac.lk

Abstract. An analytic solution for the relativistic field equations is obtained for a non-stationary, slowly rotating, cylindrically symmetric distribution of perfect fluid universe. The new metric, is regular with the exception at the point $r = 0$. There is a gravitational singularity at $r = 0$. At $t = 0$ the pressure p and density ρ are maximum and tends to ∞ throughout the radial coordinate r $(0 < r < \infty)$, but the solutions are well behaved for $t > 0$, and p and ρ are decreasing to zero as t increases through the range $0 < t < \infty$. So according to the model, it has the big bang singularity at $t = 0$, where ρ diverges.

Key words: Einstein's field equations, slow rotation, cylindrically symmetric, perfect fluid.

1. Introduction

Numerous attempts to obtain exact solutions of Einstein's field equations representing cylindrically symmetric perfect fluid matter distributions have so far been reported. But a considerable number of these solutions have not been diligent in interpreting in a physically meaningful way or even in a less acceptable way. Thus within a collection of exact solutions one finds many whose physical meaning is unknown or only partially understood.

The main reason for this may be that interpretation is difficult and uncertain. Another reason is, the observational verification of general relativity now and in the near future, is likely to depend on a very small number of exact solutions. Therefore some workers feel it is a waste of time to try to interpret metrics which have no prospect of observational verification.

But it is obvious that we cannot claim to understand general relativity unless we can determine the physics of the exact solutions we know. In the case of some of the solutions there is no physics, or the solutions do not agree with known physics. Even if this is the case it is important to try to obtain solutions and interpret them as well as we can.

Even though a good collection of solutions exist for non-rotating cases (Adler, R., Bazin, M. and Shiffer 1975, Davidson 1992, Gasperini and de Sabbata 1985, Stephani 1982), there is a rareness of the solutions for rotating cases. But during the last 40-45 years rotating objects have been studied quite extensively (Saha 1981). Some of them have studied the structure and stability of rapidly rotating fluid spheres (Butterworth and Isper 1975) with various amounts of uniform and differential rotation and some have studied uniformly rotating white dwarfs and neutron stars up to second order in angular velocity. Other papers related to numerical approach on this subject, have presented the analytic theory of slowly and uniformly rotating general relativistic bodies and discussed conditions of stability. It is very important to study objects with some kind of rotation, as almost all objects in the sky exhibit some form of rotation, and today there is even the possibility of the universe itself having a slight rotation.

2. Field equations and method of obtaining solution

The tetrad formalism (Chandrasekhar 1983) has been used to obtain the tensor components which we wanted to build-up the system of differential equations, as handling metric coefficients is rather easy in this method than in an approach such as involving the use of Euler-Lagrange equations.

We consider the non-stationary, cylindrically symmetric metric in the following general form:

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - dz^2 - tr^2(d\phi - \Omega dt)^2, \tag{1}$$

where ν, λ and Ω are functions of both time t and spatial coordinate r only. $\Omega(r,t)$ represents the dragging of inertial frames.

For slowly- rotating spacetimes, (i.e. only the first order terms in the angular velocities $\omega = \frac{d\phi}{dt}$ and Ω are considered) the following tetrad components of the Ricci tensor are obtained.

$$\begin{aligned}
R_{(00)} &= -(\nu_{rr} - \lambda_r\nu_r + \nu_r^2 + \frac{\nu_r}{r})e^{-2\lambda} - (\frac{1}{2}\frac{\nu_t}{t} + \frac{1}{4t^2} - \lambda_{tt} + \nu_t\lambda_t - \lambda_t^2)e^{-2\nu},\\
R_{(01)} &= (\frac{1}{2tr} - \frac{1}{2}\frac{\nu_r}{t} - \frac{\lambda_t}{r})e^{-(\nu+\lambda)},\\
R_{(03)} &= [\frac{1}{2}(\nu_r+\lambda_r)rt^{1/2}\Omega_r - \frac{3}{2}t^{1/2}\Omega_r - \frac{1}{2}\Omega_{rr}rt^{1/2}]e^{-(\nu+2\lambda)},\\
R_{(11)} &= -(\lambda_{tt} - \nu_t\lambda_t + \lambda_t^2 + \frac{1}{2}\frac{\lambda_t}{t})e^{-2\nu} + (\nu_{rr} - \lambda_r\nu_r + \nu_r^2 - \frac{\lambda_r}{r})e^{-2\lambda},\\
R_{(13)} &= [\frac{1}{2}(\nu_t+\lambda_t)rt^{1/2}\Omega_r - \frac{3}{4}\frac{r\Omega_r}{t^{1/2}} - \frac{1}{2}\Omega_{rt}rt^{1/2}]e^{-(2\nu+\lambda)},\\
R_{(22)} &= 0,\\
R_{(33)} &= -(\frac{1}{2}\frac{\lambda_t}{t} - \frac{1}{2}\frac{\nu_t}{t} - \frac{1}{4t^2})e^{-2\nu} - (\frac{\lambda_r}{r} - \frac{\nu_r}{r})e^{-2\lambda};
\end{aligned} \tag{2}$$

where lower indices r and t of each variable denote differentiation with respect to the spatial coordinate r and time t, and the lower indices $0,1,2,3$ represent the time coordinate t and the spatial coordinates r, z, ϕ respectively. Here and in what follows the bracketed indices denote that a tetrad frame is being used.

The matter distribution is considered as a perfect fluid with fluid pressure p and mass density ρ.

$$T_{(\mu\nu)} = (\rho + p)u_{(\mu)}u_{(\nu)} - pg_{(\mu\nu)}. \tag{3}$$

For the case of slow rotation, the tetrad components of the four-velocity are obtained as:

$$
\begin{aligned}
u^{(0)} &= 1,\\
u^{(1)} &= 0,\\
u^{(2)} &= 0,\\
u^{(3)} &= \frac{rt^{1/2}(\omega-\Omega)}{e^{\nu}}.
\end{aligned} \tag{4}
$$

With these equations, (3) can be written as:

$$
\begin{aligned}
T_{(00)} &= \rho,\\
T_{(03)} &= (p+\rho)e^{-\nu}rt^{1/2}(\Omega-\omega),\\
T_{(11)} &= p,\\
T_{(22)} &= p,\\
T_{(33)} &= p
\end{aligned} \tag{5}
$$

and T becomes,

$$T = \rho - 3p. \tag{6}$$

According to the Einstein field equation,

$$R_{(\mu\nu)} = -8\pi(T_{(\mu\nu)} - \frac{1}{2}g_{(\mu\nu)}T), \tag{7}$$

again the Ricci components are obtained as:

$$
\begin{aligned}
R_{(00)} &= -4\pi(\rho+3p),\\
R_{(01)} &= 0,\\
R_{(03)} &= -8\pi r(p+\rho)e^{-\nu}t^{1/2}(\Omega-\omega),\\
R_{(11)} &= -4\pi(\rho-p),\\
R_{(13)} &= 0,\\
R_{(22)} &= -4\pi(\rho-p),\\
R_{(33)} &= -4\pi(\rho-p).
\end{aligned} \tag{8}
$$

By using the equations (2) and (8) we get the following system of equations:

$$(\nu_{rr}-\lambda_r\nu_r+\nu_r^2+\frac{\nu_r}{r})e^{-2\lambda}+(\frac{\nu_t}{2t}-\lambda_{tt}+\nu_t\lambda_t-\lambda_t^2+\frac{1}{4t^2})e^{-2\nu} = 4\pi(\rho+3p),$$

$$(\frac{\nu_r}{2t}-\frac{1}{2tr}+\frac{\lambda_t}{r})e^{-(\nu+\lambda)} = 0,$$

$$[\frac{3}{2}t^{1/2}\Omega_r+\frac{rt^{1/2}}{2}\Omega_{rr}-\frac{(\nu_r+\lambda_r)}{2}rt^{1/2}\Omega_r]e^{-(\nu+2\lambda)}=8\pi rt^{1/2}(\rho+p)(\Omega-\omega)e^{-\nu},$$

$$(-\nu_{rr}+\lambda_r\nu_r-\nu_r^2+\frac{\lambda_r}{r})e^{-2\lambda}+(\lambda_{tt}-\nu_t\lambda_t+\lambda_t^2+\frac{\lambda_t}{2t})e^{-2\nu}=4\pi(\rho-p), \quad (9)$$

$$[\frac{3r\Omega_r}{4t^{1/2}}+\frac{\Omega_{rt}}{2}t^{1/2}r-\frac{(\nu_t+\lambda_t)}{2}rt^{1/2}\Omega_r]e^{-(2\nu+\lambda)}=0,$$

$$4\pi(\rho-p)=0,$$

$$(\frac{\lambda_r}{r}-\frac{\nu_r}{r})e^{-2\lambda}+(\frac{\lambda_t}{2t}-\frac{\nu_t}{2t}-\frac{1}{4t^2})e^{-2\nu}=4\pi(\rho-p).$$

3. Results

Assuming that ν is a function of t only and λ is a function of r and t, an analytic solution of these equations is obtained as follows.

$$e^{2\nu}=\frac{1}{k_1+\frac{4t}{k_2}} \quad (10)$$

$$e^{2\lambda}=\frac{k_2t}{r^2} \quad (11)$$

hence the metric as,

$$ds^2=\frac{1}{k_1+\frac{4t}{k_2}}dt^2-\frac{k_2t}{r^2}dr^2-dz^2-tr^2(d\phi-\Omega dt)^2. \quad (12)$$

$e^{2\lambda}\to\infty$ at $r=0\ \forall t$. Therefore there is a gravitational singularity at r=0. Furthermore, we obtained the following results.

$$p=\rho \quad (13)$$

$$\rho=\frac{k_1}{32\pi t^2} \quad (14)$$

$$\Omega=\frac{1}{(k_1+\frac{4t}{k_2})^{1/2}t}[\frac{r^5}{(1+r^2)^3}] \quad (15)$$

$$\omega=\Omega+\frac{2r^5(r^4+5r^2-20)}{k_1k_2(k_1+\frac{4t}{k_2})^{1/2}(1+r^2)^5} \quad (16)$$

where k_1 and k_2 are arbitrary constants and $k_1\geqslant 0$ and $k_2\geqslant 0$.

4. Discussion

Here in this attempt, we were interested in obtaining a family of cylindrically symmetric cosmological models for a non-stationary, slowly rotating perfect fluid distribution.

In this case, first of all, we had to choose a suitable general metric form.

Generally, in the case of non-rotating, it is convenient to adopt the metric form appropriate to cylindrical symmetry as:

$$ds^2 = D^2(r,t)dt^2 - A^2(r,t)dr^2 - B^2(r,t)dz^2 - C^2(r,t)d\phi^2. \qquad (17)$$

By using this general metric form, it has been found that a one-parameter solution of the Einstein field equations for a non-stationary, non-rotating, perfect fluid universe exists. This solution was obtained in (Davidson 1992) as:

$$\begin{aligned} D(r,t) &= (1+r^2)^{-\beta(\beta+1)/2(2\beta+1)}, \\ A(r,t) &= t^{(3\beta+1)/(7\beta+4)}(1+r^2)^{\beta(3\beta+1)/2(2\beta+1)}, \\ B(r,t) &= t^{\beta/(7\beta+4)}(1+r^2)^{\beta/2}, \\ C(r,t) &= t^{(3\beta+2)/(7\beta+4)}r(1+r^2)^{\beta/2}. \end{aligned} \qquad (18)$$

Where β is a constant.

(To secure physically acceptable models, parameter β is restricted to the range; $0 \geq \beta \geq -\frac{2}{5}$).

Considering one of the above cases, a general metric has been obtained for the case of slowly rotating in the form:

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - dz^2 - tr^2(d\phi - \Omega dt)^2 \qquad (19)$$

On the other hand we can deduce the special case $\beta = 0$ of (18), by making $\Omega = 0$ and suitably changing the arbitrary variables and constants in the sub sequent work. In this case, the geometric character of the model changes to spatial homogeneity.

In this work, we have made several assumptions and conditions (boundary conditions) to develop this metric. One of such main assumptions is that the universe has a slow rotation. It is interesting that there are plenty of physical evidences that almost every object in the sky exhibits some form of rotation, and today there is even the possibility of the universe itself being endowed with a slight rotation.

So in order to satisfy the conditions of slow rotation, only the first-order terms in the angular velocities ω and Ω have been considered.

The following boundary conditions have also been used. Since the central axis is non-rotating, ω and Ω should satisfy the boundary conditions,

$\Omega, \omega, \Omega_r, \omega_r \longrightarrow 0$ as $r \longrightarrow 0$,

In addition to this, it has been assumed that the universe is non-rotating at $r \longrightarrow \infty$. So that ω and Ω should satisfy,

$\Omega, \omega, \Omega_r, \omega_r \longrightarrow 0$ as $r \longrightarrow \infty$.

Furthermore by using equation (14) it is possible to calculate an approximate value for the constant k_1. To do this we used the data from the NASA's Wilkinson Microwave

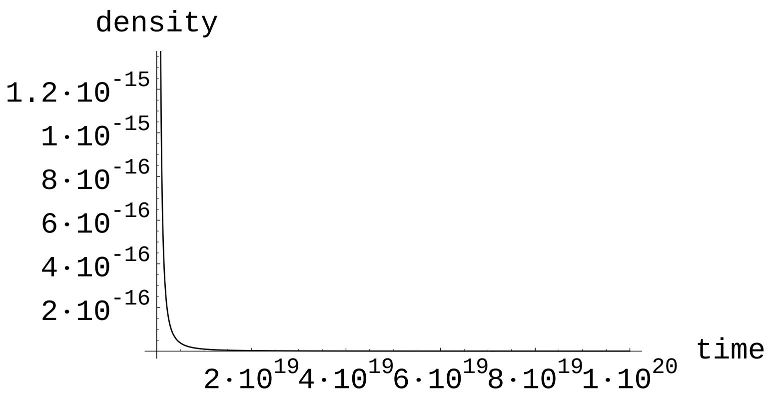

Figure 1 The variation of the density (gkm^{-3}) of the universe against time ($sec.$)

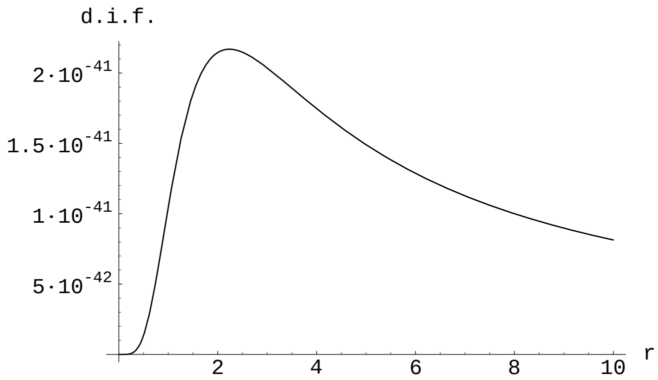

Figure 2 The behaviour of the dragging of the initial frame (Ω) against the radial coordinate (r).

Anisotropy Probe (WMAP) project. It has been estimated the age of the universe to be about 13.7 billion years old with an uncertainty of 200 million years. This measurement was made by locating the first acoustic peak in the microwave background power spectrum to determine the size of the decoupling surface. The light travel to this surface yields a reliable age for the universe. In addition to this, the lower limit of the critical density $(5 \times 10^{-15} gkm^{-3})$ Rowe (2001) was assumed as the present density of the universe. By using these two facts, an approximate value for k_1 was obtained as $8.4593298 \times 10^{33} gkm^{-1}$. Hence we were able to plot the graph of the density of the universe against time (in seconds) (See Figure 1).

By considering the equations (15) and (16), for any fixed value of time, we were able to plot the behaviour of the two angular velocities Ω (See Figure 2)and ω (See Figure 3). Here the value of the constant k_2 has been assumed as 1.

In Figure 4 we show the behaviours of the two angular velocities Ω (Left) and ω (Right) against the radial coordinate (r) and time (t).

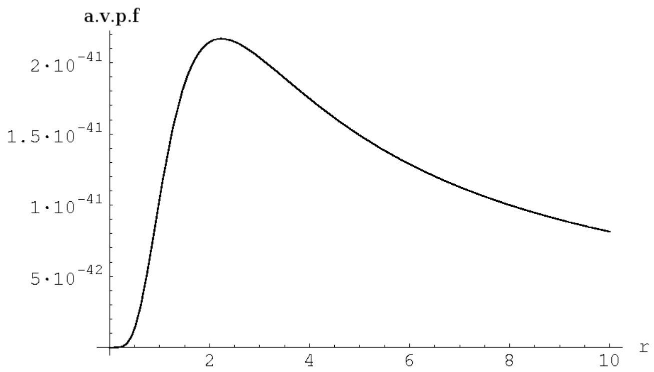

Figure 3 The behaviour of the angular velocity of the perfect fluid (ω) against the radial coordinate (r).

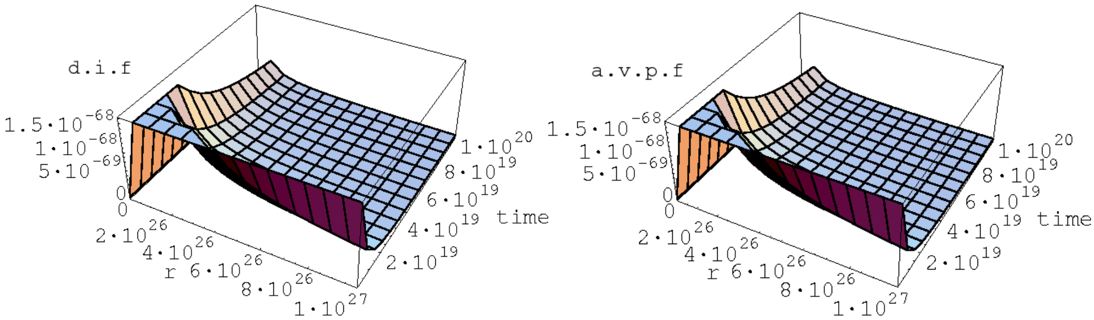

Figure 4 **Left:** shows the behaviour of the dragging of the initial frame (Ω) against the radial coordinate (r) and time (t). **Right:** shows the behaviour of the angular velocity of the perfect fluid (ω) against the radial coordinate (r) and time (t).

5. Conclusion

According to the metric (12), it is regular with the exception that at the point $r = 0$, and has a time singularity at $t = 0$ at which the pressure p and density ρ tend to ∞ throughout the radial coordinate range $0 < r < \infty$, but it is subsequently well-behaved if $k_1 = 0$, $\rho = 0$ $\forall t \neq 0$ and we may take $\rho = 0$ $\forall t$ as otherwise $\rho \to \infty$ at $t = 0$ but becomes zero in an instant!. Therefore take $k_1 \neq 0$. Further, p and ρ both are decreasing to zero as t increases through the range $0 < t < \infty$ and equations (13) and (14) imply that this fluid model has non-negative expressions for the mass density and pressure.

Even if the solution is not completely concordance with the idea of great big-bang, it makes less disagreements at the critcial points and moreover, the result here seems to agree with the physical interpretation to some extent with the big-bang theory.

Acknowledgments
It is pleasure to thank Dr. J. R. Wedagedara for his valuable support in the preparation of this paper.

References

Adler, R., Bazin, M. and Shiffer. 1975. *Introduction to General Relativity*. New York: McGraw-Hill.

Butterworth, EM., JR. Isper. 1975. . *Astrophysical Jn.* 200.

Chandrasekhar, S. 1983. *The Mathematical Theory of Black Holes*. New York: Oxford University Press.

Davidson, W. 1992. A one-parameter family of cylindrically symmetric perfect fluid cosmologies. *Phys. Rev. D* **24**.

Gasperini, M., V. de Sabbata. 1985. *Introduction to gravitation*. Singapore: World Scientific.

Rowe, E.G.P. 2001. *Geometrical Physics in Minkowski Spacetime*. London: Springer.

Saha, SK. 1981. . *Phys. Rev. D24* .

Stephani, H. 1982. *General Relativity*. Cambridge: Cambridge University Press.