RUHUNA JOURNAL OF SCIENCE
Vol. 2, September 2007, pp. 18–29
http://www.ruh.ac.lk/rjs/
ISSN 1800-279X

Testing Non-Linear Ordinal Responses in $L2 \times K$ Tables

Leslie Jayasekara
Department of Mathematics, Univerrsity of Ruhuna, Matara 81000, Sri Lanka.
leslie@maths.ruh.ac.lk
Takashi Yanagawa
Center for Biostatistics, Kurume University, Japan.
yanagawa.takashi@kurume-u.ac.jp

Abstract. In comparative studies responses of two populations are often summarized in stratified $2 \times K$ tables with ordinal categories. A test, called Q_{Et} test, is proposed for testing the homogenuity of the populations against non-linear alternatives in such tables. The asymptotic distributions of proposed test are obtained both under the null and alternative hypothesis. The powers of the Q_{Et} test and extended Mantal test are compared by simulation.

Key words: ordinal data, $L2 \times k$ tables, homogeatiy, nonlinear response, asymptotic distribution, Chi-square tests, confounding variables

1. Introduction

Data are often summarized in ordinal $2 \times K$ tables in comparative medical studies, and statistical tests such as Pearson's chi-squared test (Pearson, 1900), Wilcoxon test(Wilcoxon, 1945), Nair's test (Nair, 1986), cumulative chi-squared test (Takeuchi and Hirotsu,1982), and max χ^2 test (Hirotsu,1983) are applied to those data for detecting the difference of two distributions. It is well known that Pearson's chi-squared test has no good powers against ordered alternatives. The Wilcoxon test is specifically designed for testing location difference of two samples; also the tests are asymptotically uniformly most powerful unbiased tests for logistic linear alternatives. Whereas Nair's test is designed for detecting dispersion alternatives. The cumulative chi-squared test and max χ^2 test are ominibus tests developed for a wider class of alternatives including linear and non-linear responses. Here we call the response patterns like A, B and C in Table 1 the linear and the other patterns the non-linear; more specifically, the pattern D, E, $\cdots$, and I respectively called the $\cap$ pattern, $\cup$ pattern, $\cdots$, and $\cap\cup$ pattern. We developed the Q_t test (Jayasekara and Yanagawa, 1995; Jayasekara, Nishiyama and Yanagawa, 1999) for non-linear responses in $2 \times K$ tables. The Q_t test is shown to have higher powers than those tests just described when the control and treatment groups show the combination of the patterns of non-linear responses.

Now confounding variables such as sex, age, blood pressure and others are involved in medical data and it is important to block their effects on testing. The above statistical tests lack this function and logistic models are conventionally employed. However, as is well known, the result of logistic models depend on the goodness of fit of the models to the data,

Table 1 Response probabilities and patterns.

		Ordered categories				
	Pattern	1	2	3	4	5
A	—	0.2	0.2	0.2	0.2	0.2
B	/	0.1	0.15	0.2	0.25	0.3
C	\	0.3	0.25	0.2	0.15	0.1
D	∩	0.15	0.2	0.3	0.25	0.1
E	∪	0.25	0.2	0.1	0.15	0.3
F	∪∩	0.25	0.1	0.2	0.3	0.15
G	∩∪	0.1	0.25	0.2	0.15	0.3
H	∪∩∪	0.2	0.1	0.3	0.15	0.25
I	∩∪∩	0.15	0.25	0.1	0.3	0.2

and yet it is not easy to establish the models, in particular, when responses are non-linear and the size of the data is not large. Here we may see the raison d'etre of nonparametric tests. As far as we are aware the extended Mantel test (Mantel 1963, Lindis, Heyman, and Koch, 1978, Yanagawa 1986)(called EMT test in the sequel) is the only test that has been developed in the sprit. The EMT test adjusts for the effect of the confounding variables by stratification.

In this paper we consider the same framework as the EMT test and develop a test for testing the homogenuity against non-linear alternatives. More specifically, considering $2 \times K$ tables such as those given in Table 2 which have been constructed in the l-th stratum, $l = 1, 2, \cdots, L$, to block the effect of confounding variables, we extended the Q_t test. It is shown that the extended Q_t test has higher power in most cases than EMT test when the alternatives are non-linear.

2. The Test Statistics

We suppose in Table 2 that $\mathbf{Y}_{l1} = (Y_{l11}, Y_{l12}, \cdots, Y_{l1k})'$ and $\mathbf{Y}_{l2} = (Y_{l21}, Y_{l22}, \cdots, Y_{l2k})'$ are multinomial random vectors independently distributed with parameters $n_{l1}, (p_{l11}, p_{l12}, \cdots, p_{l1k})'$ and $n_{l2}, (p_{l21}, p_{l22}, \cdots, p_{l2k})'$ respectively ($l = 1, 2, \cdots, L$).

Suppose that categories $B_1, B_2, \cdots, B_K$ are ordinal ($B_1 < B_2 < \cdots < B_K$), and define the odds-ratio of category B_k relative to category B_1 by $\psi_{lk} = p_{l11}p_{l2k}/p_{l21}p_{l1k}$ ($k = 1, 2, \cdots, K$). The homogenuity of the distributions of the control and treatment groups in the table may be represented by $\psi_{lk} = 1$ for all $k = 1, 2, \cdots, K$ and $l = 1, 2, \cdots, L$, which we simply denote by $\psi \equiv 1$. Thus the problemma is testing $H_0 : \psi \equiv 1$ against $H_1 : \psi_{lk} \neq 1$ for some $k = 2, \cdots K$ and $l = 1, 2, \cdots, L$. In particular, considered under the alternatives are the odds ratios derived from the combinations of those linear and non-linear response patterns presented in Table 1.

We extend the Q_t test(Jayasekara and Yanagawa (1995), Jayasekara and Nishiyama(1996)) for testing H_0 vs. H_1. Let c_{lk} be the Wilcoxon scorollarye in the l-th table defined by $c_{l1} = (\tau_{l1} - N_l)/2$ and $c_{lk} = \Sigma_{j=1}^{k-1}\tau_{lj} + (\tau_{li} - N_l)/2$ for $k = 2, 3, \cdots, K$, where τ_{lk} is the marginal total in Table 2. Note that it is normalized to satisfy $\Sigma_{k=1}^{K}\tau_{lk}c_{lk} = 0$ for $l = 1, \cdots, L$.

Now for two K dimensional vectors $\mathbf{a}_l$ and $\mathbf{b}_l$ in l-th stratum we define the inner product of $\mathbf{a}_l$ and $\mathbf{b}_l$ by $(\mathbf{a}_l, \mathbf{b}_l) = \Sigma_{k=1}^{K}\tau_{lk}a_{lk}b_{lk}$ and the norm of $\mathbf{a}_l$ by $\|\mathbf{a}_l\| = (\mathbf{a}_l, \mathbf{a}_l)^{1/2}$.

Table 2 $2 \times K$ table in stratum l, $l = 1, \cdots, L$.

Stratum l	Ordered Categories B_1	B_2	$\cdots$	B_K	Total
Control	Y_{l11}	Y_{l12}	$\cdots$	Y_{l1K}	n_{l1}
Treatment	Y_{l21}	Y_{l22}	$\cdots$	Y_{l2K}	n_{l2}
Total	τ_{l1}	τ_{l2}	$\cdots$	τ_{lK}	N_l

Let c_{lk}^r be the r-th power of c_{lk}, and put $\mathbf{c}_{lr} = (c_{l1}^r, c_{l2}^r, \cdots, c_{lK}^r)'$, $r = 0, 1, \cdots, K-1$. Furthermore let $\mathbf{a}_{l0} = \mathbf{c}_{l0}/\|\mathbf{c}_{lo}\|$ and $\mathbf{a}_{lr} = \mathbf{d}_{lr}/\|\mathbf{d}_{lr}\|$, where $\mathbf{d}_{lr} = \mathbf{c}_{lr} - \Sigma_{j=0}^{r-1}(\mathbf{c}_{lr}, \mathbf{a}_{lj})\mathbf{a}_{lj}$, $r = 1, 2, \cdots, K-1$. Note that

$$(\mathbf{a}_{lr}, \mathbf{a}_{lr'}) = \begin{cases} 1 \text{ if } r = r', \\ 0 \text{ if } r \neq r', \text{for } r, r' = 0, 1, \cdots, K-1. \end{cases} \tag{1}$$

Putting for given $t \varepsilon \{1, 2, \cdots, K-1\}$

$$\begin{aligned} \mathbf{A} &= (\mathbf{a}_{lr})_{,l=1,2,\cdots,L;r=1,\cdots,t} \ (KL \times t \text{ matrix}), \\ \mathbf{Y}_2 &= (Y_{12}', \cdots, Y_{L2}')' \ (KL \text{ dimensional vector}), \\ S^2 &= \textstyle\sum_{l=1}^{L} n_{l1} n_{l2}/N_l(N_l - 1), \end{aligned}$$

and

$$\mathbf{U}_{Et} = \mathbf{A}'\mathbf{Y}_2/S$$

we propose the following Q_{Et} as a test statistic for testing H_0 vs. H_1: $Q_{Et} = \mathbf{U}_{Et}'\mathbf{U}_{Et}$ for each $t \varepsilon \{1, 2, \cdots, K-1\}$. Let U_r be the r-th elemmaent of $\mathbf{U}_{Et}$, then we have

$$U_r = \sum_{l=1}^{L} \mathbf{a}_{lr}' \mathbf{Y}_{l2}/S, \tag{2}$$

and the Q_{Et} may represented as follows:

$$Q_{Et} = U_1^2 + U_2^2 + \cdots + U_t^2.$$

Remark: Q_{Et} is identical to the test statistic of EMT test when $t = 1$, and to the Wilcoxon test statistic (Wilcoxon, 1945) when $t = 1$ and $L = 1$.

Now under H_0, the conditional distribution of $\mathbf{Y}_{l2}$ given $C_l = \{n_{l1}, n_{l2}, \tau_{l1}, \cdots, \tau_{lK}\}$ is multiple hypergeometric with

$$E[Y_{l2k}|C_l] = n_{l2}\tau_{lk}/N_l$$

$$Cov[Y_{l2k}, Y_{l2k'}|C_l] = \frac{n_{l1}n_{l2}}{N_l^2(N_l - 1)} \tau_{lk}(\delta_{jk'}N_l - \tau_{lk'}), \text{ for } k, k' = 1, \cdots, K,$$

where $\delta_{kk'} = 1$ if $k = k'$ and 0 otherwise.

THEOREM 1. *Under H_0, the elemmaents of $\mathbf{U}_{Et}$, i.e., U_r, $r = 1, 2, \cdots, t$, are uncorollaryrelated with zero mean and unit variance when conditioned on $\mathbf{C} = \{C_l, l = 1, \cdots, L\}$.*

Proof. We first show $E[\mathbf{U}_{Et}|\mathbf{C}] = 0$. Putting $\tau_l = (\tau_{l1}, \cdots, \tau_{lK})'$, we have from (1)

$$\mathbf{a}_{lr}'\tau_l = 0. \tag{3}$$

Thus

$$\begin{aligned} E[\mathbf{U}_{Et}|\mathbf{C}] &= \mathbf{A}_t' E[\mathbf{Y}_2|\mathbf{C}]/S \\ &= \mathbf{A}_t'(n_{12}\tau_1'/N_1, \cdots, n_{L2}\tau_L'/N_L)'/S \\ &= \left(\sum_{l=1}^{L} n_{l2}\mathbf{a}_{l1}'\tau_l/N_l, \cdots, \sum_{l=1}^{L} n_{l2}\mathbf{a}_{lt}'\tau_l/N_l\right)'/S \\ &= 0. \end{aligned}$$

We next compute the conditional covariance matrix of $\mathbf{U}_{Et}$. Since $\mathbf{Y}_{l2}, l = 1, \cdots, L$, are independent, the conditional covariance matrix $V(\mathbf{U}_{Et}|\mathbf{C})$ can be expressed as,

$$\begin{aligned} V(\mathbf{U}_{Et}|\mathbf{C}) &= \mathbf{A}_t' V(\mathbf{Y}_2|\mathbf{C})\mathbf{A}_t/S^2 \\ &= \mathbf{A}_t' \begin{pmatrix} V(\mathbf{Y}_{l2}'|\mathbf{C}) & & \\ & \cdot & 0 \\ 0 & \cdot & \\ & & V(\mathbf{Y}_{L2}'|\mathbf{C}) \end{pmatrix} \mathbf{A}_t/S^2 \end{aligned}$$

$$V(\mathbf{U}_{Et}|\mathbf{C}) = \left(\sum_{l=1}^{L} \mathbf{a}_{lr}' V(\mathbf{Y}_{l2}'|\mathbf{C})\mathbf{a}_{lr'}\right)/S^2 \quad \text{for } r, r' = 1, \cdots, t. \tag{4}$$

Since

$$\sum_{l=1}^{L} \mathbf{a}_{lr}' V(\mathbf{Y}_{l2}'|\mathbf{C})\mathbf{a}_{lr'} = \sum_{l=1}^{L} \frac{n_{l1}n_{l2}}{N_l^2(N_l-1)}[N_l \mathbf{a}_{lr}' \begin{pmatrix} \tau_{l1} & & \\ & \cdot & 0 \\ 0 & \cdot & \\ & & \tau_{lk} \end{pmatrix} \mathbf{a}_{lr'} - \mathbf{a}_{lr}'\tau_l\tau_l'\mathbf{a}_{lr'}],$$

it follows from (3) that

$$\sum_{l=1}^{L} \mathbf{a}_{lr}' V(\mathbf{Y}_{l2}'|\mathbf{C})\mathbf{a}_{lr'} = \sum_{l=1}^{L} \frac{n_{l1}n_{l2}}{N_l(N_l-1)}(\mathbf{a}_{lr}, \mathbf{a}_{lr'}).$$

Thus from (1)

$$\sum_{l=1}^{L} \mathbf{a}_{lr}' V(\mathbf{Y}_{l2}'|\mathbf{C})\mathbf{a}_{lr'}/S^2 = \begin{cases} 1 & \text{if } r = r', \\ 0 & \text{if } r \neq r', r, r' = 0, 1, \cdots, K-1. \end{cases}$$

Therefore from (4), we have

$$V(\mathbf{U}_{Et}|\mathbf{C}) = \mathbf{I}_t.$$

3. Asymptotic Distributions

Theorem 1 shows that the elemmaents of $\mathbf{Q}_{Et}$ are uncorollaryrelated and furthermore from (2) they are linear combinations of $\mathbf{Y}_{l2} = (Y_{l21}, \cdots, Y_{l2K})$. However, their weight vectors, $\mathbf{a}_{lr}$'s, depends on N_l, which makes the asymptotic theory not straightforward. We assume that when $N_l \to \infty$ the marginal totals n_{li} and τ_{lk} for $l = 1, \cdots L$, satisfy:
(**A**1) $n_{li}/N_l \to \gamma_{li}$, $0 < \gamma_{li} < 1$, for $i = 1, 2$, and $\tau_{lk}/N_l \to \rho_{lk}$, $0 < \rho_{lk} < 1$, for $k = 1, 2, \cdots, K$. To begin with we review the normal approximation of a multiple hypergeometric distribution.

3.1. Normal Approximation of a Multiple Hypergeometric Distribution

Plackett (1981) showed that when assumption (A1) is satisfied the asymptotic conditional distribution of $\mathbf{X}_l = (Y_{l22}, \cdots, Y_{l2K})'$ given $C_l = \{n_{l1}, n_{l2}, \tau_{l1}, \cdots, \tau_{lK}\}$, is a $K-1$ dimensional normal with mean $\mathbf{m}_{l2}$ and covariance matrix V_l, where $\mathbf{m}_{l2} = (m_{l22}, \cdots, m_{l2k})'$ and $V_l^{-1} = (\sigma_{ljk})$ with $\sigma_{lkk'} = m_{l11}^{-1} + m_{l21}^{-1} + (m_{l1k}^{-1} + m_{l2k}^{-1})\delta_{kk'}$, for $k, k' = 2, \cdots, K$ and $l = 1, \cdots, L$. Here the sequence $\{m_{lik}\}$, $i = 1, 2$; $k = 1, 2, \cdots, K$, is determined uniquely by equations $\sum_{k=1}^{K} m_{lik} = n_{li}$, $\sum_{i=1}^{2} m_{lik} = \tau_{lk}$, and $m_{l11}m_{l2k}/m_{l21}m_{l1k} = \psi_{lk}$, for $i = 1, 2$; $k = 1, 2, \cdots, K$ and $l = 1, \cdots, L$. It is known (Sinkhorn, 1967) that the sequence may be obtained by the following iterative scaling procedure:

$$
\begin{aligned}
m_{l1k}^{(1)} &= \frac{n_{l1}}{K},\ k = 1, 2, \cdots, K \\
m_{l21}^{(1)} &= \frac{n_{l2}}{K[1 + \sum_{j=2}^{K}(\psi_{lj} - 1)/K]} \\
m_{l2k}^{(1)} &= \frac{n_{l2}\psi_{lk}}{K[1 + \sum_{j=2}^{K}(\psi_{lj} - 1)/K]},\ k = 2, \cdots, K \\
m_{lik}^{(2)} &= \frac{m_{lik}^{(1)}\tau_{lk}}{m_{l.k}^{(1)}}, \\
m_{lik}^{(3)} &= \frac{m_{lik}^{(2)}n_{li}}{m_{li.}^{(2)}}, \\
&\vdots \\
m_{lik}^{(2h)} &= \frac{m_{lik}^{(2h-1)}\tau_{lk}}{m_{l.k}^{(2h-1)}}, \\
m_{lik}^{(2h+1)} &= \frac{m_{lik}^{(2h)}n_{li}}{m_{li.}^{(2h)}},\ h = 1, 2, \cdots, \text{and } l = 1, \cdots, L.
\end{aligned}
$$

3.2. Asymptotic Distributions Under H_0

We first evaluate the weight, a_{lrk}. We write $N_l^{1/2}a_{lrk} = O(1)$ if and only if $N_l^{1/2}a_{lrk}$ tends to a constant as $N \to \infty$.

LEMMA 1. *If (A1) is satisfied, then*

(i) $N_l^{-1}c_{lrk} = O(1)$, *where* $c_{lrk} = c_{lk}^r$, *is the r-th power of the k-th Wilcoxon scorollarye in the l-th table, for* $r = 1, 2, \cdots, K-1$, $k = 1, 2, \cdots, K$ *and* $l = 1, \cdots, L$.

(ii) Let a_{l0k} *be the k-th elemmaent of* $\boldsymbol{a}_{l0}$. *Then* $N_l^{-r}(\boldsymbol{c}_{lr}, \boldsymbol{a}_{l0})a_{l0k} = O(1)$, *for* $r = 1, 2, \cdots, K-1$, $k = 1, 2, \cdots, K$ *and* $l = 1, \cdots, L$.

(iii) Let $d_{l\nu k}$ *be the k-th component of* $\boldsymbol{d}_{l\nu}$. *If* $N_l^{-\nu}d_{l\nu k} = O(1)$, $k = 1, 2, \cdots, K$, *then for any* $\nu = 1, 2, \cdots$, *we have*

(a) $N_l^{-2\nu-1}\|\boldsymbol{d}_{l\nu}\|^2 = O(1)$,

(b) $N_l^{-r}(\boldsymbol{c}_{lr}, \boldsymbol{d}_{l\nu})d_{l\nu k}/\|\boldsymbol{d}_{l\nu}\|^2 = O(1)$, $l = 1, \cdots, L$.

(iv) $N_l^{-r}d_{lrk} = O(1)$ *for* $r = 1, 2, \cdots, K-1$, $k = 1, 2, \cdots, K$ *and* $l = 1, \cdots, L$.

(v) $N_l^{1/2}a_{lrk} = O(1)$ *for* $r = 1, 2, \cdots, K-1$, $k = 1, 2, \cdots, K$ *and* $l = 1, \cdots, L$.

Proof. (i) By the definition of c_{lk}, and from (A1), we may get $N_l^{-1}c_{lk} = O(1)$ for $l = 1, \cdots, L$. Thus it is obvious that $N_l^{-r}c_{lk}^r = O(1)$. (ii) By the definition of $\mathbf{a}_{l0}$ we have $a_{l0k} =$

$1/N_l^{1/2}$ for all k. So from (i) we obtain $N_l^{-(r+1/2)}(\mathbf{c}_{lr}, \mathbf{a}_{l0}) = O(1)$. Thus we have (ii). (iii) (a) The result may be obtained by the definition of $\mathbf{d}_{lv}$. (b) Expanding the inner product (c_{lr}, d_{lv}) and applying (i) we may show $N_l^{-(r+l+1)}(c_{lr}, d_{lv}) = O(1)$. Now using (a), the result follows. (iv) To prove this result we use induction on r. In case of $r = 1$,

$$d_{l1k} = c_{l1k} - (\mathbf{c}_{l1}, \mathbf{a}_{l0})a_{l0k}, \quad \text{for } k = 1, 2, \cdots, K.$$

Applying (i) and (ii), it follows that $N_l^{-1}d_{l1k} = O(1)$ for $k = 1, 2, \cdots, K$. Suppose that the result is true for $r = 1, 2, \cdots, m-1$. Since

$$\begin{aligned} \mathbf{d}_{lm} &= \mathbf{c}_{lm} - \sum_{j=0}^{m-1}(\mathbf{c}_{lm}, \mathbf{a}_{lj})\mathbf{a}_{lj}, \\ &= \mathbf{c}_{lm} - (\mathbf{c}_{lm}, \mathbf{a}_{l0})\mathbf{a}_{l0} - \sum_{j=1}^{m-1}(\mathbf{c}_{lm}, \mathbf{d}_{lj})\frac{\mathbf{d}_{lj}}{\|\mathbf{d}_{lj}\|^2}, \end{aligned}$$

it follows that $N_l^{-m}d_{lmk} = O(1)$ from (i), (ii) and (iii). So the result is true for $r = m$. Thus by the induction the result follows. (v) From the definition of $\mathbf{a}_{lr}$ and also by (iv) the result is straightforward.

Next, we consider the asymptotic distribution of the test statistics under H_0. To apply the normal approximation in section 3.1 we represent the t dimensional vector $\mathbf{U}_{Et}$ by:

$$\mathbf{U}_{Et} = \mathbf{B}'\mathbf{W}/S, \tag{5}$$

where
$\mathbf{B} = (\mathbf{b}_{lr}), \mathbf{b}_{lr} = (a_{lr2} - a_{lr1}, \cdots, a_{lrK} - a_{lr1})'N_l^{1/2}$, $l = 1, \cdots, L$; $r = 1, 2, \cdots, t$,
$\mathbf{W} = (W_1', W_2', \cdots, W_L')'$, $W_l = N_l^{-1/2}(\mathbf{X}_l - n_{l2}\tau_l/N_l)$.

THEOREM 2. *Under H_0, Q_{Et} is asymptotically distributed as a chi-squared distribution with t degrees of freedom as $N \to \infty$, $l = 1, 2, \cdots, L$.*

Proof. From section 3.1 we have $m_{lik} = n_{li}\tau_{lk}/N_l$, under H_0. Thus the conditional distribution of W_i given $C_l = \{n_{l1}, n_{l2}, \tau_{l1}, \cdots, \tau_{lK}\}$ converges in distribution to $N_{K-1}(0, \sum_{l0})$ as $N_l \to \infty$, where $\sum_{l0}^{-1} = (\sigma_{ljk0})$, $j, k = 2, \cdots, K$, with $\sigma_{ljk0} = [\rho_{l1}^{-1} + \delta_{jk}\rho_{lk}^{-1}]/(\gamma_{l1}\gamma_{l2})$. Furthermore, since $N_l^{1/2}a_{lrk} = O(1)$ from Lemmama 1(v), we have $\mathbf{b}_{lr} = O(1)$. Thus as $N_l \to \infty$, $l = 1, \cdots, L$, it will be easy to show that $\mathbf{U}_{Et} = \mathbf{B}'\mathbf{W}/S$ converges in distribution to a t dimensional normal distribution with mean zero and the covariance matrix

$$V[\mathbf{U}_{Et}]_\infty = \mathbf{B}' \begin{pmatrix} \sum_{10} & & \\ & \cdot & \mathbf{0} \\ \mathbf{0} & \cdot & \\ & & \sum_{L0} \end{pmatrix} \mathbf{B}/S^2 \tag{6}$$

Now putting

$$M_l = \begin{pmatrix} \rho_{l2}(N_l - \rho_{l2}) & -\rho_{l2}\rho_{l3} & \cdots & -\rho_{l2}\rho_{lK} \\ -\rho_{l3}\rho_{l2} & \rho_{l3}(N_l - \rho_{l3}) & \cdots & -\rho_{l3}\rho_{lK} \\ \cdot & \cdot & \cdots & \cdot \\ -\rho_{lK}\rho_{l2} & -\rho_{lK}\rho_{l3} & \cdots & \rho_{lK}(N_l - \rho_{lK}) \end{pmatrix} \gamma_{l1}\gamma_{l2},$$

we may show

$$M_l \sum_{l0}^{-1} = \mathbf{I}_{K-1}.$$

Furthermore, from (1)

$$\mathbf{B}' \begin{pmatrix} M_1 & & \\ & \cdot & \mathbf{0} \\ \mathbf{0} & \cdot & \\ & & M_L \end{pmatrix} \mathbf{B}/S^2 \sim \mathbf{I}_t,$$

where $\sim$ means that the ratio of the both hands side tends to one as $N_l \to \infty$, $l = 1, 2, \cdots, L$. Thus from (6)

$$V[\mathbf{U}_{Et}]_\infty \sim \mathbf{I}_t,$$

and $Q_{Et} = \mathbf{U}'_{Et}\mathbf{U}_{Et}$ follows asymptotically a chi-squared distribution with t degrees of freedom.

3.3. Asymptotic Distribution Under Contiguous Alternatives

In this section we obtain the asymptotic distribution of Q_{Et} under alternative hypothesis $H_1 : \psi_{lk} = 1 + A_{lk}/N_l^{1/2}$, for $k = 2, 3, \cdots, K$, where A_{lk} is a constant.

LEMMA 2. *Under H_1, we may represent $m_{lik} = m^0_{lik} + N_l^{1/2}\eta_{lik} + O(N_l^{1/2})$ for $i = 1, 2$, $k = 1, 2, \cdots, K$ and $l = 1, \cdots, L$, where $m^0_{lik} = n_{li}\tau_k/N_l$ is the asymptotic mean under H_0, and*

$$\eta_{i1} = (-1)^{i+1}N_l^{1/2}\gamma_{l1}\gamma_{l2}\rho_{l1}\sum_{j=2}^{K}(\psi_{lj} - 1)\psi_{lj}$$
$$\eta_{ik} = (-1)^{i}N_l^{1/2}\gamma_{l1}\gamma_{l2}\rho_{lk}[\psi_{lk} - 1 - \sum_{j=2}^{K}(\psi_{lj} - 1)\psi_{lj}]$$

$k = 2, 3, \cdots, K$.

Proof. Adopting the iterative scaling algoritheorem in section 3.1, we have the following expressions for $m^{(1)}_{l1k}, m^{(1)}_{l21}m^{(1)}_{l2k}, m^{(2)}_{li1}$ and $m^{(2)}_{lik}$, under H_1.

$$m^{(1)}_{l1k} = \frac{n_{l1}}{K}$$
$$m^{(1)}_{l21} = \frac{n_{l2}}{K}[1 - \sum_{j=2}^{K}\frac{(\psi_{lj} - 1)}{K} + o(N_l^{-1/2})]$$
$$m^{(1)}_{l2k} = \frac{n_{l2}}{K}[\psi_{lk} - \sum_{j=2}^{K}\frac{(\psi_{lj} - 1)}{K} + o(N_l^{-1/2})],\ k = 2, 3, \cdots, K,$$
$$m^{(2)}_{li1} = m^0_{li1} + (-1)^{i+1}N_l\gamma_{l1}\gamma_{l2}\rho_{l1}\sum_{j=2}^{K}\frac{(\psi_{lj} - 1)}{K} + o(N_l^{-1/2}),$$
$$m^{(2)}_{lik} = m^0_{lik} + (-1)^{i}N_l\gamma_{l1}\gamma_{l2}\rho_{lk}[\psi_{lk} - 1 - \sum_{j=2}^{K}\frac{(\psi_{lj} - 1)}{K}] + o(N_l^{-1/2}).$$

Using mathematical induction on v, we may show

$$m_{lik}^{(v)} = m_{lik}^{0} + N_l^{1/2}\eta_{lik} + o(N_l^{1/2}),$$

$k = 1, 2, \cdots, K$, and $v = 3, 4, \cdots$. Thus we have the desired results.

THEOREM 3. *Under H_1, Q_{Et} is asymptotically distributed as a non-central chi-squared distribution with t degrees of freedom. The noncentrality parameter is given by $\lambda = \sum_{r=1}^{t} \delta_r^2$, where $\delta_r = \sum_{l=1}^{L} N_l\gamma_{l1}\gamma_{l2} \sum_{k=2}^{K} a_{lrk}\rho_{lk}(\psi_{lk} - 1)/S$.*

Proof. From Section 3.1 and Lemmama 2 it follows that under H_1, the conditional distribution of W_l given $C_l = \{n_{l1}, n_{l2}, \tau_{l1}, \cdots, \tau_{lK}\}$ converges in distribution to $N_{K-1}(\eta_{l2}, \Sigma_{l0})$, where $\eta_{l2} = (\eta_{l22}, \cdots, \eta_{l2K})'$, and Σ_{l0} is that given in the proof of Theorem 2. Thus under H_1, $\mathbf{U}_{Et} = \mathbf{B}'\mathbf{W}/S$ converges in distribution to t dimentional normal distribution with mean

$$\delta_{Et} = \mathbf{B}'(\eta_{12}', \cdots, \eta_{L2}')'/S$$

and covariance matrix $V[\mathbf{U}_{Et}]_\infty$, which is shown to be $\mathbf{I}_t$ in the proof of Theorem 2. The r-th elemmaent of δ_{Et}, say δ_r, is obtained as:

$$\delta_r = \sum_{l=1}^{L}\sum_{k=2}^{K} N_l^{1/2}(a_{lrk} - a_{lr1})\eta_{l2k}/S.$$

From (7) and $\psi_{l1} = 1$, we have

$$\delta_r = \sum_{l=1}^{L} N_l\gamma_{l1}\gamma_{l2} \sum_{k=2}^{K} a_{lrk}\rho_{lk}(\psi_{lk} - 1)/S.$$

The theorem is immediately obtained from these results.

COROLLARY 1. *The power of U_r^2 is approximately maximized when ln $\psi_{lk} = \beta_l a_{lrk}$, $k = 1, 2, \cdots, K$, for some constant β_l, $l = 1, 2, \cdots, L$.*

Proof. From the proof of Theorem 3 it follows that U_r^2 follows asymptotically a noncentral chi-squared distribution with one degree of freedom with noncentral parameter δ_r^2. Thus the asymptotic power of U_r^2 for testing H_0 vs. H_1 may be approximated by

$$P(U_r^2 \geq \chi_1^2(\alpha)|H_0) \approx \Phi(\delta_r - \chi(\alpha)),$$

where Φ is the cdf of a standard normal distribution. Since δ_r may be represented by $\delta_r = \sum_{l=1}^{L} \gamma_{l1}\gamma_{l2}(a_{lr}, \psi_l - 1)/S$, this power is maximized when $\psi - 1 = \beta_l \mathbf{a}_{lr}$, that is when ln $\psi_{lk} \approx \beta_l \mathbf{a}_{lrk}$ for some constant β_l.

From the corollaryollary the statistic $Q_{Et} = U_1^2 + U_2^2 + \cdots + U_t^2$ is viewed as a sum of the statistics that are asymptotically optimum against the alternatives which are expressed as log linearities of the odds ratios with scorollarye a_{lrk}, the standardized r-th power of the Wilcoxon scorollarye.

4. Simulation Studies

Simulation was conducted to compare the Q_{Et}, $t = 1,2,3,4$, test with the EMT test (Mantel 1963, Landis, Heyman and Koch, 1978, Yanagawa 1986). Because the EMT test with Wilcoxon scorollarye is equivalent to the Q_{E1} test, we herein considered the EMT test with scorollaryes $0,1,2,\cdots$, and $K-1$ assigned to categories $B_1, B_2, \cdots$, and B_K, respectively.

First we assessed Type I error of the Q_{Et}, $t = 1,2,3,4$ and EMT tests at the significance level $\alpha = 0.05$. The response probabilities employed are those listed in Table 1. We considered four strata and combinations of response patterns shown in the first column of Table 3. For example, (⟋, ∩, ∽, ∽∪) in the table means that the response probabilities in the 1st stratum are $p_{111} = p_{121} = 0.1$, $p_{112} = p_{122} = 0.15$, $p_{113} = p_{123} = 0.2$, $p_{114} = p_{124} = 0.25$, $p_{115} = p_{125} = 0.3$; 2nd stratum are $p_{211} = p_{221} = 0.1$, $p_{212} = p_{222} = 0.15$, $p_{213} = p_{223} = 0.2$, $p_{214} = p_{224} = 0.25$, $p_{215} = p_{225} = 0.3$; and so on. We generated 10,000, four 2×5 tables for each combination of patterns and computed empirical significance levels when $n_{l1} = n_{l2} = 60, 80$, and 100. The results are listed in Table 3. The table shows that Type I error of the Q_{Et} and EMT tests are close to the nominal level for all combinations of patterns.

Second we assessed the powers of the Q_{Et}, $t = 1,2,3,4$, and EMT tests. We conducted similar simulation as above by using again the response probabilities listed in Table 1. Considering the combinations of pattern of distribution of $\mathbf{Y}_1$ from {(—, —, —, —), (⟋, ⟋, ⟋, ⟋), (⟍, ⟍, ⟍, ⟍), $\cdots$, (∽∿, ∽∿, ∽∿, ∽∿), (⟋, ∩, ∽, ∽∪), (⟍, ∪, ∿, ∿∽), (∩, ∽, ∽∿, ∿∽), (∪, ∽, ∿, ∿∽)}

we computed the powers of the tests for all combinations of patterns of each distribution, 48 all together, when $n_{l1} = n_{l2} = 100$, $l = 1,2,3$ and 4. The tests which give the largest and second largest powers are listed in Table 4a, 4b, and 4c. For example, the entry of the 2nd row and 3rd column in Table 4a means that when the pattern of $\mathbf{Y}_1$ is and that of $\mathbf{Y}_2$ is the test with the largest power is Q_{E4} followed by Q_{E3}; and the entry of the 2nd row and 4th column in Table 4c means that when the pattern of $\mathbf{Y}_1$ is and that of $\mathbf{Y}_2$ then the test with the largest power is $\mathbf{Q}_{E4}$ followed by Q_{E3}. The tests in the tables show that those tests have equal powers. The tables show that in most combinations, 45 among 48, the powers of the class of the Q_{Et} test are larger or equal to than those of the EMT test. Table 5 lists the maximum, mean and minimum values of the powers of each test for 48 combinations of response patterns considered in Table 4. Inspection of the table shows that the mean and minimum powers of the Q_{Et} test dominates the corollaryresponding values of the other tests, and that the maximum powers of the tests are almost equal.

5. Discussion

The Q_{Et} test is proposed for testing the homogeneity against non-linear responses in $L2 \times K$ tables. We took into account the combinations of patterns of linear and non-linear responses summarized in Table 1, and shown that the class of Q_{Et} test is superior to the extended Mantel test (Mantel 1963, Landis, Heyman and Koch 1978, Yanagawa 1986). Those non-linear patterns we considered often appear, for example, in Phase III randomized clinical trials for proving the efficacy of a new drug against the active control,in which the efficacy is sometimes categorized as excellent, effective slightly effective, not effective and aggravation. We emphasize that in such example, the response probabilities like 0.15, 0.25, 0.1, 0.3 and

Table 3 Estimated Type I errors of the Q_{Et}, $t = 1, 2, 3, 4$, and extended Mantel test (EMT).

Pattern	Sample size	Estimated Type I error levels				
	$n_{l1} = n_{l2}$, $l = 1, 2, 3, 4$	Q_{E1}	Q_{E2}	Q_{E3}	Q_{E4}	EMT
(—, —, —, —)	60	0.052	0.052	0.052	0.052	0.052
	80	0.05	0.048	0.048	0.047	0.051
	100	0.049	0.049	0.0.05	0.051	0.049
(╱, ╱, ╱, ╱)	60	0.054	0.052	0.052	0.049	0.054
	80	0.051	0.051	0.048	0.047	0.049
	100	0.052	0.052	0.051	0.05	0.053
(╲, ╲, ╲, ╲)	60	0.053	0.054	0.053	0.05	0.052
	80	0.05	0.049	0.05	0.05	0.053
	100	0.052	0.051	0.049	0.048	0.05
(∩, ∩, ∩, ∩)	60	0.052	0.051	0.049	0.049	0.051
	80	0.049	0.048	0.05	0.05	0.049
	100	0.052	0.047	0.05	0.05	0.052
(∪, ∪, ∪, ∪)	60	0.054	0.056	0.052	0.052	0.054
	80	0.051	0.051	0.05	0.048	0.051
	100	0.052	0.052	0.051	0.051	0.052
(∪∩, ∪∩, ∪∩, ∪∩)	60	0.053	0.054	0.052	0.053	0.052
	80	0.05	0.051	0.051	0.051	0.049
	100	0.052	0.051	0.05	0.048	0.052
(∩∪, ∩∪, ∩∪, ∩∪)	60	0.052	0.05	0.051	0.052	0.053
	80	0.049	0.049	0.05	0.047	0.049
	100	0.051	0.052	0.051	0.05	0.052
(∪∩∪, ∪∩∪, ∪∩∪, ∪∩∪)	60	0.054	0.054	0.052	0.051	0.053
	80	0.052	0.049	0.051	0.05	0.052
	100	0.051	0.052	0.051	0.05	0.052
(∩∪∩, ∩∪∩, ∩∪∩, ∩∪∩)	60	0.052	0.05	0.051	0.049	0.053
	80	0.05	0.05	0.05	0.049	0.05
	100	0.05	0.051	0.053	0.054	0.05
(╱, ∩, ∪∩, ∪∩∪)	60	0.053	0.054	0.051	0.05	0.053
	80	0.052	0.049	0.049	0.05	0.052
	100	0.053	0.052	0.054	0.05	0.055
(╲, ∪, ∩∪, ∩∪∩)	60	0.054	0.052	0.051	0.048	0.053
	80	0.05	0.05	0.049	0.047	0.051
	100	0.049	0.048	0.049	0.049	0.048
(∩, ∪∩, ∪∩∪, ∩∪∩)	60	0.053	0.053	0.049	0.051	0.053
	80	0.05	0.051	0.051	0.05	0.049
	100	0.05	0.049	0.05	0.047	0.05
(∪, ∪∩, ∩∪, ∩∪∩)}	60	0.055	0.051	0.048	0.05	0.054
	80	0.05	0.051	0.05	0.05	0.049
	100	0.052	0.05	0.049	0.048	0.051

Table 4 Tests which give the largest and second largest powers:

$\mathbf{Y}_1$	$\mathbf{Y}_2$: (∕,∩,∪∩,∪∩∪)	(∖,∪,∩∪,∩∪∩)	(∩,∪∩,∪∩∪,∩∪∩)	(∪,∪∩,∩∪,∩∪∩)
(—,—,—,—)	Q_{E4}, Q_{E3}	Q_{E4}, Q_{E3}	Q_{E3}, Q_{E4}	Q_{E4}, Q_{E2}
(∕,∕,∕,∕)	Q_{E4}, Q_{E1}	(Q_{E1}, Q_{E2}, Q_{E3}, Q_{E4}, EMT)	(Q_{E1}, Q_{E2}, Q_{E3}, Q_{E4}, EMT)	EMT, Q_{E2}
(∖,∖,∖,∖)	(Q_{E1}, Q_{E2}, Q_{E3}, Q_{E4}, EMT)	(Q_{E1}, Q_{E2}, Q_{E3}, Q_{E4}, EMT)	(Q_{E1}, Q_{E2}, Q_{E3}, Q_{E4}, EMT)	(Q_{E1}, Q_{E2}, Q_{E3}, Q_{E4}, EMT)
(∩,∩,∩,∩)	Q_{E2}, Q_{E4}	(Q_{E2}, Q_{E3}, Q_{E4}, EMT)	Q_{E2}, Q_{E3}	(Q_{E2}, Q_{E3}, Q_{E2}, Q_{E1})
(∪,∪,∪,∪)	(Q_{E2}, Q_{E3}, Q_{E4}, EMT)	Q_{E2}, Q_{E3}	(Q_{E2}, Q_{E3}, Q_{E4}, Q_{E1})	Q_{E3}, Q_{E2}
(∪∩,∪∩,∪∩,∪∩)	Q_{E3}, Q_{E4}	(Q_{E3}, Q_{E4}, Q_{E2})	(Q_{E3}, Q_{E4}, Q_{E2})	Q_{E4}, Q_{E3}
(∩∪,∩∪,∩∪,∩∪	(Q_{E3}, Q_{E4}, Q_{E2})	Q_{E4}, Q_{E3}	(Q_{E3}, Q_{E4}, EMT)	Q_{E3}, Q_{E4}
(∪∩∪,∪∩∪,∪∩∪,∪∩∪)	Q_{E4}, Q_{E3}	Q_{E4}, Q_{E3}	Q_{E4}, Q_{E3}	Q_{E4}, Q_{E3}
(∩∪∩,∩∪∩,∩∪∩,∩∪∩)	Q_{E4}, Q_{E3}	Q_{E4}, Q_{E3}	Q_{E4}, Q_{E3}	Q_{E4}, Q_{E2}
(∕,∩,∪∩,∪∩∪)	-	Q_{E4}, Q_{E3}	Q_{E4}, Q_{E1}	Q_{E4}, Q_{E3}
(∖,∪,∩∪,∩∪∩)	Q_{E4}, Q_{E3}	-	Q_{E4}, Q_{E3}	EMT, Q_{E3}
(∩,∪∩,∪∩∪,∩∪∩)	Q_{E4}, Q_{E1}	Q_{E1}, Q_{E2}	-	Q_{E4}, Q_{E3}
(∪,∪∩,∩∪,∩∪∩)	Q_{E4}, Q_{E3}	EMT, Q_{E1}	Q_{E4}, Q_{E3}	-

Table 5 The maximum, mean and the minimum powers of the tests for 48 combinations of the patterns in Table 4.

	Q_{E1}	Q_{E2}	Q_{E3}	Q_{E4}	EMT
Max.	1	1	1	1	1
Mean	0.343	0.561	0.705	**0.832**	0.345
Min.	0.049	0.076	0.084	**0.154**	0.048

0.2,i.e. pattern is not unreasonable. It is suggested in the simulation that when all combinations of those response patterns are taken into account the Q_{E4} test is good choice. The Q_{Et} is shown to be the sum of U_r^2, $r = 1, 2, \cdots, t$, that are asymptotically optimum against the alternatives which are expressed as log linearities of the odds ratios with scorollarye a_{lrk}, the standardized r-th power of the Wilcoxon scorollarye.

References

Pearson, K. (1900), On criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling, Philos. Mag., series 5, 50, 157-175. (Reprinted 1948 in Karl Pearsons Early statistical papers, ed. By E. S. Pearson, Cambridge University Press.)

Nair, V. N. (1986), On Testing in Industrial Experiments with Ordered Categorical Data, Journal of American Statistical Association, 58, 690-700.

Takeuchi, K. and Hirotsu, C. (1982), The Cumulative Chi Square Method Against Ordered Alternative in Two-way Contingency Tables, Reports of Statistical Application Research, Japanese Union of Scientists and Engineers, 29, 1-13.

Hirotsu, C. (1983), Defining the pattern of association in two-way contingency tables, Biometrika, 70, 579-589.

Jayasekara, L. and Yanagawa, T. (1995) Testing ordered categorical data in $2 \times k$ Tables by the statistics with orthonormal scorollarye vectors, *Bulletin of Informatics and Cybernetics*, **27**, 129-145.

Jayasekara, L., Nishiyama, H. and Yanagawa, T. (1999) Versatile Test for Ordinal Data in $2 \times k$ Tables,

Lindis, J.R., Heyman, E.R. and Koch, G.G. (1978), Average partial association in three way contingency tables: a review and discussion of alternative tests, *International Statistical Review*, **46**, 237-254.

Mantel, N. (1963), Chi-square tests with one degree of freedom: extensions of the Mantel-Haenszel procedure, *Journal of American Statistical Association*, **58**, 690-700.

Plackett, R.L. (1981). The Analysis of Categorical Data, London: Griffin.

Sinkhorn, R. (1967), Diagonal equivalence to matrices with prescribed row and column sums, *Amer. Math. Month.*, 74, 402-5.

Wilcoxon, Frank (1945), Individual Comparisons by Ranking Methods, *Biometrics* **1**, 80-83.

Ynagawa, T. (1986), Analysis of Discrete Multivariate Data, Tokyo: Kyoritu Shuppan.