RUHUNA JOURNAL OF SCIENCE
Vol. 1, September 2007, pp. 41–47
http://www.ruh.ac.lk/rjs/
ISSN 1800-279X

Slowly rotating dust sphere for free space in general relativity with uniform matter distribution

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Abstract. Einstein field equations for a charged dusty universe have already investigated. In this paper we present a new class of analytical solutions in terms of canonical coordinates for Einstein's field equations; assuming that the spacetime is spherically symmetric, formed by non-charged dust with uniform matter distribution. The metric we considered is of the form, $ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta(d\phi - \Omega dt)^2$, where ν, λ and Ω are functions of the radial coordinate r only. Our model has only a space singularity at $r = 0$ and the solutions are well behaved for $r > 0$. In addition, we assume that the proper density ρ is *constant*. $\Omega(r)$, the angular velocity of the inertial frame; can be an arbitrary function of r, which satisfies required boundary conditions to be a slow rotation.

Key words: spherical symmetry, slowly rotation, free-space

1. INTRODUCTION

The general relativity theory still dwells at the highest place among the gravitational theories; but the physical meaning of many exact solutions of Einstein's equations are unknown, or only partially understood Bonnor (1992). One of the major difficulties is the, either long time taken to find a experimental verifications to prove the solutions have real physical meaning. However general relativity is central to the understanding of frontier astrophysical phenomena such as black holes, pulsars, the big bang and the universe itself Milner (2000), Ltartle (2003). It is interesting to note that every object in the space exhibits some form of rotation. For further details see, Bayin (1981). Therefore, during the last decades, rotating objects have been studied quite extensively Tiwari et al. (1986), Bayin (1981). In this paper our aim is to study the case of slowly rotating dust sphere in free space. We consider two types of rotations in our equation, $\Omega(r)$, which represents the dragging of inertial frames and ω, which represents the angular velocity of dust distribution, along the coordinate axis ϕ, i.e., $\omega = d\phi/dt$. In the approximation of slow rotation (1) leads to a system of equation in the first order terms, and then new analytical solutions for λ, ν, Ω, and ω are found subjected to the assumptions $\rho =$ *constant* and time independent. The boundary conditions for Ω are $\lim_{r\to\infty}\Omega(r) \to 0$, $\lim_{r\to\infty}(\partial\Omega(r)/\partial r) \to 0$ and $\lim_{r\to 0}\Omega(r) \to 0$, $\lim_{r\to 0}(\partial\Omega(r)/\partial r) \to 0$. The same boundary conditions should be satisfied by ω. We take the cosmological constant Λ to be equal to zero for simplicity and consider a static metric.

2. FIELD EQUATIONS

We consider the static, spherically symmetric metric in the standard form Chandrasekher (1910)

$$ds^2 = g_{ij}dx^i dx^j; \quad i,j = 0,1,2,3$$

$$\begin{aligned} ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - r^2 d\theta^2 \\ - r^2 \sin^2\theta(d\phi - \Omega dt)^2; \end{aligned} \tag{1}$$

where ν, λ and Ω are functions of the radial coordinate r only. The coordinates x^0, x^1, x^2 and x^3 correspond to t, r, θ and ϕ respectively. The function Ω represents the angular velocity of the inertial frames along the rotation axis.
The Einstein field equations are, Chandrasekher (1910), Tiwari et al. (1986),

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij}.$$

and Ricci tensor R_{ij} is defined by Tiwari et al. (1986) ,

$$R_{ij} = g^{kl}R_{ikjl}. \tag{2}$$

The Ricci tensor components in tetrad form for slowly rotating sphere, related with the metric (1) are as follows:

$$\begin{aligned} R_{(00)} &= -e^{-2\lambda}(\nu_{rr} - \lambda_r\nu_r + \nu_r^2 + \frac{2\nu_r}{r}), \\ R_{(11)} &= e^{-2\lambda}(\nu_{rr} - \lambda_r\nu_r + \nu_r^2 - \frac{2}{r}\lambda_r), \\ R_{(22)} &= (\frac{\nu_r}{2} - \frac{\lambda_r}{r} + \frac{1}{r^2})e^{-2\lambda} - \frac{1}{r^2}, \\ R_{(03)} &= -\frac{1}{2}\sin\theta e^{-(2\lambda+\nu)}[\Omega_{rr}r + 4\Omega_r - \Omega_r r(\lambda_r + \nu_r)]; \end{aligned} \tag{3}$$

where, Ω_r and ω_r denote differentiation with respect to r.

The energy momentum tensor $T_{(ij)}$ of dust is defined by Tiwari et al. (1986),

$$T_{(ij)} = \rho U_{(i)}U_{(j)};$$

where ρ is the proper density of the universe, and U^i are the four velocity components of the matter distribution along the each coordinate axis. The components U^i are given by,

$$\begin{aligned} U^0(r) &= \frac{dx^0}{ds}, \\ U^1(r) &= U^2(r) = 0, \\ U^3(r) &= \frac{d\phi}{dx^0}\frac{dx^0}{ds} = \omega U^0. \end{aligned}$$

We use the tetrad frame $e^{i}_{(\alpha)}$ which is associated with the metric (1),

$$\begin{aligned} e^{\alpha}_{(0)} &= (e^{-\nu}, 0, 0, 0), \\ e^{\alpha}_{(1)} &= (0, e^{-\lambda}, 0, 0), \\ e^{\alpha}_{(2)} &= (0, 0, \frac{1}{r}, 0), \\ e^{\alpha}_{(3)} &= (0, 0, 0, \frac{1}{r}\sin\theta). \end{aligned}$$

$$e^{i}_{(\alpha)} e_{(\beta)i} = \eta_{(\alpha\beta)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$e^{i}_{(\alpha)} e^{(\alpha)}_{j} = \delta^{i}_{j}$$

(an index within parentheses denotes a tetrad index).

For the case of slowly rotation, tetrad components for four velocity are obtained as follows,

$$U_{(0)} = 1, \quad U_{(1)} = 0, \quad U_{(2)} = 0, \quad U_{(3)} = re^{-\nu}\sin\theta(\Omega - \omega).$$

The field equations for dust universe in the tetrad frame can be expressed in the form

$$R_{(ij)} = -8\pi(T_{(ij)} - \frac{1}{2}g_{(ij)}T). \tag{4}$$

The tetrad components for energy momentum tensor are as follows:

$$\begin{gathered} T_{(00)} = \rho \\ T_{(01)} = T_{(02)} = T_{(11)} = T_{(12)} = T_{(22)} = T_{(13)} = T_{(33)} = 0, \\ T_{(03)} = \rho e^{-\nu} r\sin\theta(\Omega - \omega). \end{gathered} \tag{5}$$

From (4)and (5) the Ricci tensor components can be obtained as follows:

$$\begin{gathered} R_{(01)} = R_{(02)} = R_{(12)} = R_{(13)} = R_{(23)} = 0, \\ R_{(00)} = R_{(11)} = R_{(22)} = R_{(33)} = -4\pi\rho, \\ R_{(03)} = -8\pi\rho r\sin\theta(\Omega - \omega). \end{gathered} \tag{6}$$

From (3)and (6) the following non- linear system of equations can be obtained,

$$\begin{aligned} 4\pi\rho &= e^{-2\lambda}(\nu_{rr} - \lambda_r\nu_r + \nu_r^2 + \frac{2\nu_r}{r}), \\ -4\pi\rho &= e^{-2\lambda}(\nu_{rr} - \lambda_r\nu_r + \nu_r^2 - \frac{2}{r}\lambda_r), \\ -4\pi\rho &= (\frac{\nu_r}{2} - \frac{\lambda_r}{r} + \frac{1}{r^2})e^{-2\lambda} - \frac{1}{r^2}, \\ 8\pi\rho\sin\theta(\Omega - \omega) &= \frac{1}{2}\sin\theta e^{-(2\lambda+\nu)}(\Omega_{rr}r + 4\Omega_r - \Omega_r r(\lambda_r + \nu_r)). \end{aligned} \tag{7}$$

3. THE SOLUTION

All the solutions are obtained for $\rho = constant$. The solutions for $e^{2\lambda}$ and $e^{2\nu}$ can be obtained by solving the first three of system of equations (7). We have,

$$e^{2\lambda} = \left(\frac{3}{6+8\pi\rho r^2}\right) \quad e^{2\nu} = 4\left(\frac{(12+2kr^2)^{5/4}}{r^{3/2}}\right); \quad k = 8\pi\rho. \tag{8}$$

We have to assume a value for Ω which is a function of r only that satisfies the given boundary conditions. Therefore, Ω is arbitrary and in this paper we choose functions for Ω as a function of r only. Let us take,

$$\Omega(1) = \frac{r^3}{(2+r^2)^4} + \frac{r(4r^2+r^3)}{(r^2+1)^5}.$$

Then $\omega(r)$ can be found as follows,

$$\omega = \Omega - \frac{e^{-(2\lambda+\nu)}}{2k}\left[\Omega_{rr}r + 4\Omega_r - \Omega_r r(\lambda_r + \nu_r)\right]. \tag{9}$$

It is clear that both $\Omega(r)$ and $\omega(r)$ satisfy the boundary conditions as $r \to 0$ and $r \to \infty$.

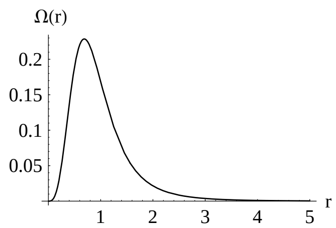

Figure 1 variation of dragging velocity of inertial frames in spacetime with increasing r (r, the distance from origin of the universe to the imaginated position); the dragging inertial frames have a maximal velocity in $0 < r < 1$ (r, measured in light years)

Fig. 1 shows the Ω has a maximum value $0 \leqslant r < 1$, meaning that the inertial frames drag with a maximum velocity in $0 \leqslant r < 1$.

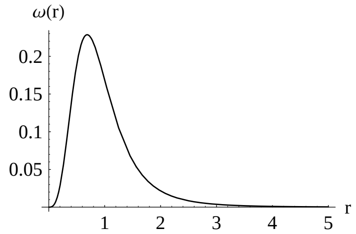

Figure 2 The variation of angular velocity of coordinate axis ϕ, with increasing r (r, distance from origin of the universe to the imaginary point)

We choose a different Ω as follows,

$$\Omega(2) = \frac{r^3}{(2r^5+4)^2} \sin\left(\frac{4r^2}{(r^2+5)^2}\right).$$

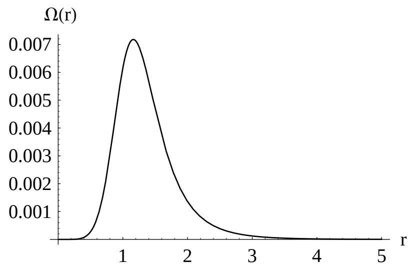

Figure 3 variation of dragging velocity of inertial frames in spacetime with increasing r, shows one maximum rotation $0 < r < 1$, value of it less than to Ω_1

Shape of the graph similar to that of fig. 1, but the maximum value for rotations in a different range and the value is completely different from that is given in Ω_1.

Using (9) we can find new ω for Ω_2 .

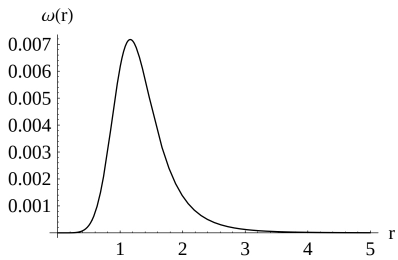

Figure 4 The variation of angular velocity of axis ϕ, with increasing r, shows maximum rotation $1 < r < 2$

By substituting value $e^{2\lambda}$, $e^{2\nu}$ and Ω in (1) we can obtain two different models for the universe as follows:

$$ds^2 = 4\left(\frac{(12+2kr^2)^{5/4}}{r^{3/2}}\right)dt^2 - \left(\frac{3}{6+8\pi\rho r^2}\right)dr^2 - r^2 d\theta^2 \\ - r^2\sin^2\theta\left(d\phi - \left(\frac{r^3}{(2+r^2)^4} + \frac{r(4r^2+r^3)}{(r^2+1)^5}\right)dt^2\right)^2.$$

$$ds^2 = 4\left(\frac{(12+2kr^2)^{5/4}}{r^{3/2}}\right)dt^2 - \left(\frac{3}{6+8\pi\rho r^2}\right)dr^2 - r^2 d\theta^2 \\ - r^2\sin^2\theta\left(d\phi - \frac{r^3}{(2r^5+4)^2}\sin\left(\frac{4r^2}{(r^2+5)^2}\right)dt^2\right)^2.$$

4. DISCUSSION AND CONCLUSION

In our work, we like to obtain a family of spherical symmetric cosmological models for a non-stationary rotating dust distribution. It is a fact that the motions of the planets were investigated theoretically to a greater degree of accuracy, the theory of the motion of a rigid body was developed and applied to the problem of the rotation of the earth. However, applications of general relativity are few indeed, a state of affairs may perhaps be accounted for by its extreme mathematical complexity; investigators have found specific problems mathematically difficult and have soon turned away in discouragement. We developed our metric for the particular case when $\rho = constant$, for uniform matter distribution. To satisfy the slow rotation condition we consider terms of the first order in Ω. We make the following observations.

1. The model has a space singularity at $r = 0$
2. Infinite number of solutions can be found for Ω and ω. Considering two expressions for Ω the dragging velocity of the inertial frames, we find that it maximixed at at least one time $0 \leqslant r < \infty$. The angular velocity of dust distribution along the axis ϕ, also has one minimum value in $0 \leqslant r < \infty$. Although, the solutions are satisfy the boundary conditions. All the solutions shows maximum and minimum rotations which make us wonder because according to our assumptions no external force has been exist exception the gravity, and it is rather difficult to give high accuracy physical explanations to solutions. To sum up, solutions for Einstein field equations are not unique and linear, it is not easy to give physical explanations.

5. ACKNOWLEDGMENTS

We are thanking to Dr.J.R.Wedegadera making valuable comments and suggestions.

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